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ON INSTABILITY IN THE PRESENCE OF SEVERAL RESONANCES

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The equilibrium position stability of an autonomous system of ordinary differential equations is considered in the case of n pairs of pure imaginary roots with the simultaneous presence of several resonances. It is shown using Chetaev's theorem [1] that when among the solutions of the model system there are increasing solutions of the invariant ray type, the complete system is Liapunov unstable.

Let us consider the system

$$x_*^{\cdot} = Ax_* + X_*(x_*), \quad X_*(0) = 0 \tag{1}$$

where x_* and X_* are $2n$-dimensional vectors of space E_{2n}; A is a constant square matrix with pure imaginary eigenvalues $\pm i\omega_s$ ($\omega_s > 0$, $s = 1, \ldots, n$) among which there are no multiples, and $X_*(x_*)$ is a holomorphic vector function of x_* whose expansion in powers of x_* begins with an m-th order form.

Let system (1) have $\mu > 1$ resonance relations of the form

$$\langle \Omega, P_\nu \rangle = 0, \quad \nu = 1, \ldots, \mu \tag{2}$$
$$\Omega = (\omega_1, \ldots, \omega_q), \quad P_\nu = (p_{\nu 1}, \ldots, p_{\nu q})$$
$$|P_\nu| = \sum_{j=1}^{q} |p_{\nu j}| = k, \quad k = m + 1 \geqslant 3$$

where P_ν is a vector of dimension q ($q \leqslant n$) with integral mutually disjoint components, and k is an odd number.

The stability of equilibrium position of the autonomous system (1) with condition (2) was investigated in [2 – 6] in the first nonlinear order. Below we consider the equilibrium position stability of the complete system (1) when condition (2) is satisfied.

Using the special linear transform it is possible to reduce system (1) to the form

$$x^{\cdot} = i\omega x + X(x, y), \quad y^{\cdot} = -i\omega y + Y(x, y) \tag{3}$$

where x and y are complex conjugate n-dimensional vectors; ω is a diagonal $n \times n$ matrix, and $X(x, y)$ and $Y(x, y)$ are holomorphic complex conjugate n-dimensional vector functions whose expansions in powers of x and y begin with m-th order forms.

Using the nonlinear normalizing transform we can reduce system (3) in polar coordinates r_s, φ_s ($s = 1, \ldots, n$) [6] to the form (equations for φ_α are omitted)

$$r_j^{\cdot} = 2 \sum_{\nu=1}^{\mu} R_\nu Q_{\nu j}(\theta_\nu) + \Upsilon_j(r, \varphi), \quad r_\alpha^{\cdot} = \Upsilon_\alpha(r, \varphi) \tag{4}$$

$$\theta_\nu^{\cdot} = \sum_{i=1}^{\mu}\sum_{j=1}^{q} \frac{|p_{\nu j}|}{r_j} R_i Q_{ij}'(\theta_i) + \Theta_\nu(r, \varphi)$$
$$j = 1, \ldots, q; \quad \nu = 1, \ldots, \mu; \quad \alpha = q + 1, \ldots, n$$
$$R_\nu^2 = \prod_{l=1}^{q} r_l^{|p_{\nu l}|}, \quad \theta_\nu = \sum_{j=1}^{q} p_{\nu j}\varphi_j$$
$$Q_{\nu j}(\theta_\nu) = a_{\nu j}\cos\theta_\nu + b_{\nu j}\sin\theta_\nu, \quad Q_{\nu j}' = dQ_{\nu j} / d\theta_\nu$$
$$r = (r_1, \ldots, r_n), \quad \varphi = (\varphi_1, \ldots, \varphi_n), \quad \Theta_\nu(r, \varphi) \sim O(\| r \|^{(k-1)/2})$$
$$\Upsilon_s(r, \varphi) \sim O(\| r \|^{(k+1)/2}), \; s = 1, \ldots, n$$
$$Q_{\nu j}(\theta_\nu) \equiv 0, \quad \text{if} \quad p_{\nu j} = 0$$

In the corresponding model system

$$\Upsilon_s(r, \varphi) \equiv 0, \;\; \Theta_\nu(r, \varphi) \equiv 0 \;\; (s = 1, \ldots, n; \; \nu = 1, \ldots, \mu)$$

For the model system to have an increasing solution of the invariant ray type

$$r_j = k_j b(t), \quad k_j > 0, \quad b^{\cdot} = 2b^{k/2}, \; j = 1, \ldots, q \qquad (5)$$
$$\theta_\nu = \theta_\nu^\circ = \text{const}, \; \nu = 1, \ldots, \mu$$

it is necessary and sufficient that

$$k_j = \sum_{\nu=1}^{\mu} R_\nu^\circ Q_{\nu j}^\circ > 0, \quad \sum_{i=1}^{\mu}\sum_{j=1}^{q} \frac{|p_{\nu j}| R_i^\circ Q_{ij}^{\circ\prime}}{k_j} = 0 \qquad (6)$$
$$R_\nu^\circ = R_\nu(k_1, \ldots, k_q), \quad Q_{\nu j}^\circ = Q_{\nu j}(\theta_\nu^\circ);$$
$$j = 1, \ldots, q; \;\; \nu = 1, \ldots, \mu$$

Indeed, by substituting the solution of form (5) into the model system and setting

$$b^{\cdot} = 2b^{k/2} \qquad (7)$$

we obtain the required relations (6). It is, on the other hand, evident that the solution of form (5) of the model system exists, if $k_j > 0$ $(j = 1, \ldots, q)$ and θ_ν° $(\nu = 1, \ldots, \mu)$ satisfying (6) can be found. Function $b(t)$ is then obtained from Eq. (7).

We introduce the notation ($\delta_{\beta h}$ is the Kronecker delta)

$$A_{\beta h} = \sum_{\nu=1}^{\mu} S_{\nu\beta}^\circ K_{\nu h} - 2\delta_{\beta h}, \quad A_{\beta, n+\nu} = S_{\nu\beta}^{\circ\prime}$$
$$A_{n+\nu, \beta} = \sum_{i=1}^{\mu} (T_{\nu i}^{\circ\prime} K_{i\beta} - L_{\nu i\beta}), \quad A_{n+\nu, n+i} = -T_{\nu i}^\circ$$
$$K_{\nu\beta} = \frac{1}{2\sqrt{q-\beta}}\Big[\sum_{l=\beta+1}^{q} |p_{\nu l}| - (q-\beta)|p_{\nu\beta}|\Big]$$
$$L_{\nu i\beta} = \frac{R_i^\circ}{\sqrt{q-\beta}}\Big[\sum_{l=\beta+1}^{q} \frac{|p_{\nu l}| Q_{il}^{\circ\prime}}{k_l} - \frac{(q-\beta)|p_{\nu\beta}| Q_{i\beta}^{\circ\prime}}{k_\beta}\Big]$$

$$S_{\nu\beta}(\theta_\nu) = \frac{2R_\nu^\circ}{(q-\beta+1)\sqrt{q-\beta}}\Big[\sum_{l=\beta+1}^{q}\frac{Q_{\nu l}(\theta_\nu)}{k_l} - \frac{(q-\beta)\,Q_{\nu\beta}(\theta_\nu)}{k_\beta}\Big]$$

$$\overset{\circ}{S}_{\nu\beta} = S_{\nu\beta}(\theta_\nu^\circ)$$

$$T_{\nu i}(\theta_i) = R_i^\circ \sum_{j=1}^{q}\frac{|p_{\nu j}|\,Q_{ij}(\theta_i)}{k_j}, \quad T_{\nu i}^\circ = T_{\nu i}(\theta_i^\circ)$$

$$\beta, h = 1, \ldots, q-1; \quad \nu, i = 1, \ldots, \mu$$

T h e o r e m. If we assume

$$\det \| A_{\upsilon\zeta} - N\delta_{\upsilon\zeta} \| \neq 0, N = 1, 2, \ldots \; (\upsilon, \zeta = 1, \ldots, n+\mu; \quad \upsilon, \zeta \neq q, \ldots, n) \tag{8}$$

and that the respective model system has an increasing solution of the invariant ray type (5), then the equilibrium position of the complete system (4) is Liapunov unstable.

P r o o f. We introduce in (4) the generalized n-dimensional cylindrical coordinates ρ, ψ_β $(\beta = 1, \ldots, q-1)$, r_α $(\alpha = q+1, \ldots, n)$ defined by formulas

$$r_1 = k_1\rho\cos\psi_1; \quad r_j = k_j\rho\cos\psi_j \prod_{l=1}^{j-1}\sin\psi_l, \quad j = 2, \ldots, q-1 \tag{9}$$

$$r_q = k_q\rho\prod_{l=1}^{q-1}\sin\psi_l; \quad r_\alpha = r_\alpha, \quad \alpha = q+1, \ldots, n$$

The following values correspond to the increasing solution of form (5) in the coordinate system (9):

$$\psi_\beta = \psi_\beta^\circ, \quad \cos\psi_\beta^\circ = (q-\beta+1)^{-1/2}, \quad \sin\psi_\beta^\circ = \left(\frac{q-\beta}{q-\beta+1}\right)^{1/2}$$

$$\beta = 1, \ldots, q-1$$

We linearize the new system with respect to variables ψ_β, θ_ν in the neighborhood of point ψ_β°, θ_ν° taking into account conditions (6), apply the transformation

$$\bar\psi_\beta = \psi_\beta^* + \sum_{l=1}^{2(1+\gamma)} c_{\beta l}\rho^{l/2}, \qquad \beta = 1, \ldots, q-1$$

$$\bar\theta_\nu = \theta_\nu^* + \sum_{l=1}^{2(1+\gamma)} d_{\nu l}\rho^{l/2}, \qquad \nu = 1, \ldots, \mu$$

where $c_{\beta l}$ and $d_{\nu l}$ are some constants and γ is a parameter defined below and, allowing for (8), obtain a system of the form

$$\rho^\cdot = 2\varkappa\rho^{k/2} + F(r_*, \psi^*, \varphi)$$

$$\psi_\beta^{*\cdot} = \varkappa\rho^{k/2-1}\Big(\sum_{h=1}^{q-1} A_{\beta h}\psi_h^* + \sum_{i=1}^{\mu} A_{\beta,\, n+i}\theta_i^*\Big) + F_\beta(r_*, \psi^*, \varphi)$$

$$\theta_\nu^{*\cdot} = \varkappa\rho^{k/2-1}\Big(\sum_{h=1}^{q-1} A_{n+\nu,\, h}\psi_h^* + \sum_{i=1}^{\mu} A_{n+\nu,\, n+i}\theta_i^*\Big) + F_{n+\nu}(r_*, \psi^*, \varphi)$$

$$r_\alpha^{\cdot} = F_\alpha (r_*, \psi^*, \varphi)$$
$$\beta = 1, \ldots, q-1;\ \nu = 1, \ldots, \mu;\ \alpha = q+1, \ldots, n$$
$$r_* = (\rho, r_{q+1}, \ldots, r_n),\ \ \psi^* = (\psi_1^*, \ldots, \psi_{q-1}^*),\ \ \theta^* = (\theta_1^*, \ldots, \theta_\mu^*),\ \ \varkappa = q^{(2-k)/4}$$

$$F(r_*, \psi^*, \varphi) = F^{(1)}(r_*, \psi^*, \varphi) + \rho^{k/2} F^{(2)}(r_*, \psi^*, \varphi)$$
$$F^{(1)} \sim O(\| r_* \|^{(k+1)/2}),\ \ F^{(2)}(0, \psi^*, \varphi) \sim O(\| (\psi^*, \theta^*) \|)$$
$$F_\upsilon (r_*, \psi^*, \varphi) = F_\upsilon^{(1)}(r_*, \psi^*, \varphi) + \rho^{-1/2} F_\upsilon^{(2)}(r_*, \psi^*, \varphi) + \rho^{k/2-1} F_\upsilon^{(3)}(r_*, \psi^*, \varphi)$$
$$F_\upsilon^{(1)} \sim O(\| r_* \|^{(k-1)/2}),\ \ \ F_\upsilon^{(2)} \sim O(\| r_* \|^{(k+1)/2}),$$
$$F_\upsilon^{(3)}(0, \psi^*, \varphi) \sim O(\| (\psi^*, \theta^*) \|^2)$$
$$F_\upsilon(\rho, 0, \ldots, 0) \sim O(\rho^{(k+1)/2+\gamma}),\ \ \upsilon = 1, \ldots, n+\mu$$
$$(\upsilon \neq q, \ldots, n)$$
$$F_\alpha(r_*, \psi^*, \varphi) \sim O(\| r_* \|^{(k+1)/2}),\ \ \alpha = q+1, \ldots, n$$

Let us consider functions

$$V = \rho,\ \ \ W_\beta = \psi_\beta^{*2} - \rho^{2(1+\gamma)}$$
$$W_{n+\nu} = \theta_\nu^{*2} - \rho^{2(1+\gamma)},\ \ \ \ W_\alpha = r_\alpha^2 - \rho^{2(1+\gamma)}$$
$$\beta = 1, \ldots, q-1;\ \nu = 1, \ldots, \mu;\ \alpha = q+1, \ldots, n$$

The inequality $VV^{\cdot} > 0$ is satisfied in the cone K_1 containing the increasing solution of the model system for $0 < \| r_* \| < \tau$ (τ is fairly small). The cone K_2 is determined by the inequality

$$\max_\iota W_\iota \leqslant 0,\ \ \iota = 1, \ldots, n+\mu\ (\iota \neq q)$$

Continuing our reasoning in conformity with [7] (Theorem 3.1), we determine with an accuracy to terms of order $\rho^{1/2+\sigma}$ the derivatives

$$W_{\upsilon 0}^{\cdot} = 2\varkappa\rho^\sigma \Big[\sum_{\zeta=1}^{n+\mu} A_{\upsilon\zeta}\delta_\zeta - 2(1+\gamma)\Big]$$
$$W_{\alpha 0}^{\cdot} = -4\varkappa\rho^\sigma (1+\gamma);\ \ \ \ \sigma = 2\gamma + k/2 + 1,\ \ |\delta_\zeta| \leqslant 1$$
$$\upsilon, \zeta = 1, \ldots, n+\mu\ \ (\upsilon, \zeta \neq q, \ldots, n);\ \ \alpha = q+1, \ldots, n$$

It is obvious that for all admissible values of δ_ζ, and fairly large γ and when $\rho < \tau$ we have

$$W_{\iota 0}^{\cdot} < 0,\ \ \ \iota = 1, \ldots, n+\mu\ (\iota \neq q)$$

Hence functions V and $W = \max W_\iota$ ($\iota = 1, \ldots, n+\mu;\ \iota \neq q$) satisfy Chetaev's theorem on instability [1]. The theorem is proved.

E x a m p l e. Let us consider the interaction of two resonances

$$\omega_1 + \omega_2 - \omega_3 = 0,\ \ 2\omega_1 - \omega_4 = 0$$

of which the first is strong and the second weak (in the terminology of [5]). Let the model system be in this case of the form

$$r_1^{\cdot} = 2b_{11}\sqrt{r_1 r_2 r_3}\sin\theta_1 + 2b_{21}\sqrt{r_1^2 r_4}\sin\theta_2 \tag{10}$$
$$r_\gamma^{\cdot} = 2b_{1\gamma}\sqrt{r_1 r_2 r_3}\sin\theta_1\ \ \ \ (\gamma = 2, 3),\ \ \ \ \ r_4^{\cdot} = 2b_{24}\sqrt{r_1^2 r_4}\sin\theta_2$$

$$\theta_1^{\cdot} = \left(\frac{b_{11}}{r_1} + \frac{b_{12}}{r_2} + \frac{b_{13}}{r_3}\right)\sqrt{r_1r_2r_3}\cos\theta_1 + \frac{b_{21}}{r_1}\sqrt{r_1^2r_4}\cos\theta_2$$
$$\theta_2^{\cdot} = \frac{2b_{11}}{r_1}\sqrt{r_1r_2r_3}\cos\theta_1 + \left(\frac{2b_{21}}{r_1} + \frac{b_{24}}{r_4}\right)\sqrt{r_1^2r_4}\cos\theta_2$$
$$\theta_1 = \varphi_1 + \varphi_2 - \varphi_3, \quad \theta_2 = 2\varphi_1 - \varphi_4$$
$$(\mathrm{sign}\ b_{1j}b_{1h} = 1\ (j, h = 1, 2, 3);\ \mathrm{sign}\ b_{21}b_{24} = -1)$$

which has the following increasing solution:

$$r_1 = |b_{11}b_{12}b_{13}|\, b(t), \quad r_4 = |b_{11}|\, b_{24}^2 b(t)$$
$$r_\gamma = |b_{1\gamma}|\,(|b_{12}b_{13}| + |b_{21}b_{24}|)b(t), \quad \gamma = 2, 3$$
$$\theta_\nu = (-1)^{\nu-1}(\pi/2)\,\mathrm{sign}\, b_{\nu 1}, \quad \nu = 1, 2$$

Consequently, in conformity with the proved theorem, the equilibrium position of the complete system, to which corresponds the model system (10), is Liapunov unstable for all nonzero parameters $b_{\nu j}$, except those that satisfy the condition $|b_{21}b_{24} / b_{12}b_{13}| = N$ $(N = 2, 3, \ldots)$.

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